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Cycles in a synchronous neural network

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Abstract. We extend the standard calculation of the number of metastable states (fixed points) so as to take into account cyclic attractors as well. We calculate analytically the average number of cycles in the Little model as a function of the fraction of flipping spins $(1 - q)/2$. We find that the cycles of length two are by far the most common type of attractors. In particular, for small values of the storage parameter, the inverse cycles ($q = -1$) are the dominant attractors.

The most popular neural network model of associative memory is probably the Hopfield model (Hopfield 1982). In this model the neurons are modelled by spins $S_i = \pm 1, i = 1, \ldots, N$ that evolve according to the neural dynamics

$$S_i(t+1) = \mathrm{sign}\left(\sum_j J_{ij} S_j(t) \right) \qquad i = 1, \ldots, N \tag{1}$$

with the synaptic couplings J_{ij} given by the Hebb rule

$$J_{ij} = \frac{1}{N} \sum_{l=1}^{P} \xi_i^l \xi_j^l. \tag{2}$$

Here the P binary patterns $\boldsymbol{\xi}^l = (\xi_1^l, \ldots, \xi_N^l)$, $l = 1, \ldots, P$ are the items one wants to store in the network. It is usually assumed that the components ξ_i^l are randomly chosen as ± 1 with equal probability, and that the number of patterns scales linearly with N, i.e. $P = \alpha N$. The retrieval of a stored pattern will depend on its proximity to the initial state $\boldsymbol{S}(t = 0)$, hence the denomination of content-addressable memory for this type of information storage system.

The update in the Hopfield model is asynchronous or sequential, i.e. the spins are updated one after the other in either a fixed or variable order. This fact, together with the symmetry of the synaptic couplings, implies that the neural dynamics minimizes the energy function $E = -\frac{1}{2} \sum_{ij} J_{ij} S_i S_j$, and hence that its stationary states are fixed points only (Hopfield 1982). These results have allowed the use of the powerful tools of the equilibrium statistical mechanics to investigate the nature of the stationary states of the Hopfield model (Amit *et al* 1987). An interesting alternative to the sequential update is the synchronous or parallel update, where all neurons are updated simultaneously. This leads to the Little model (Little 1974) which, in the case of symmetric couplings, has limit cycles of length two in addition to fixed points (Peretto 1984, Frumkin and Moses 1986). Moreover, as pointed out by Peretto (1984), the parallel dynamics minimizes the energy

$E = -\sum_i |\sum_j J_{ij} S_i|$, thus allowing the use of statistical mechanics techniques to study the equilibrium properties of the model (Fontanari and Köberle 1988).

Besides the standard equilibrium statistical mechanics approach mentioned above, which is restricted to neural networks with symmetric couplings, there is a powerful technique that can be used to study analytically networks with non-symmetric couplings as well, namely, the counting of the average number of metastable states, i.e. states that are stable to all single spin flips (Bray and Moore 1980, Gardner 1986, Treves and Amit 1988). However, up to now this method has been only applied to the study of sequential dynamics, which is understandable since the fixed points are the same for both types of update. In this note we extend that method so as to take into account the periodic attractors as well. More specifically, we calculate the average number of cycles of length two of the Little model as a function of the storage parameter α and the fraction of flipping spins $(1-q)/2$.

As in the thermodynamic calculation, to investigate the stationary states of the Little model we need to duplicate the configuration space (Fontanari and Köberle 1988). Thus, two states $\boldsymbol{S}$ and $\boldsymbol{\sigma}$ belong to a cycle of length two if they satisfy simultaneously the inequalities

$$S_i \sum_j J_{ij}\sigma_j > 0 \qquad \text{and} \qquad \sigma_i \sum_j J_{ij} S_j > 0 \tag{3}$$

for $i = 1, \ldots, N$. The generalization of these constraints to describe cycles of larger length is straightforward. The average number of cycles of length two with a fixed fraction $(1-q)/2$ of flipping spins is thus given simply by

$$\langle \mathcal{N}_q \rangle = \left\langle \sum_{\boldsymbol{S}} \sum_{\boldsymbol{\sigma}} \delta\left(Nq, \sum_i^N S_i\sigma_i\right) \left[\prod_i^N \Theta\left(S_i \sum_j J_{ij}\sigma_j\right) \Theta\left(\sigma_i \sum_j J_{ij} S_j\right)\right]\right\rangle \tag{4}$$

where $\delta(m, n)$ is the Kronecker delta, and $\Theta(x) = 1$ if $x > 0$ and 0 otherwise. Here the notation $\langle \ldots \rangle$ stands for the average over the set of stored patterns. For $q = 1$ we recover the calculation of the number of fixed points (Gardner 1986, Amit and Treves 1988), while for $q = -1$ we have cycles in which the states are inverse of each other (mirror states). We note that the inverse cycles were the only ones detected in the thermodynamic calculation (Fontanari and Köberle 1988).

As the calculations are straightforward and rather unilluminating we will only sketch it in the sequel. The first step is to extract the terms on J_{ij} from the arguments of the theta functions. This can be done with the aid of delta functions, yielding:

$$\begin{aligned} \langle \mathcal{N}_q \rangle = \int_{-\infty}^{\infty} \prod_i^N \frac{\mathrm{d}x_i\,\mathrm{d}\tilde{x}_i}{2\pi} \Theta(x_i) \mathrm{e}^{\mathrm{i}x_i\tilde{x}_i} \int_{-\infty}^{\infty} \prod_i^N \frac{\mathrm{d}y_i\,\mathrm{d}\tilde{y}_i}{2\pi} \Theta(y_i) \mathrm{e}^{\mathrm{i}y_i\tilde{y}_i} \int_{-\pi}^{\pi} \frac{\mathrm{d}\tilde{q}}{2\pi} \mathrm{e}^{\mathrm{i}Nq\tilde{q}} \\ \times \left\langle \sum_{\boldsymbol{S}} \sum_{\boldsymbol{\sigma}} \exp\left[-\mathrm{i}\tilde{q} \sum_i S_i\sigma_i - \mathrm{i} \sum_{ij} J_{ij}(\tilde{x}_i S_i \sigma_j + \tilde{y}_i \sigma_i S_j)\right]\right\rangle \end{aligned} \tag{5}$$

where we have also used the integral representation of the Kronecker delta. The average over the stored patterns can be easily carried out by introducing the auxiliary parameters $Nm^l = \sum_i S_i \xi_i^l$ and $Nn^l = \sum_i \sigma_i \xi_i^l$. After performing the averages, we introduce the following saddle-point parameters: $NA_x = \sum_i \tilde{x}_i^2$, $NA_y = \sum_i \tilde{y}_i^2$, $NB_x = \sum_i \tilde{x}_i$, $NB_y = \sum_i \tilde{y}_i$, $NC_x = \sum_i \tilde{x}_i S_i \sigma_i$, $NC_y = \sum_i \tilde{y}_i S_i \sigma_i$, $ND = \sum_i \tilde{x}_i \tilde{y}_i S_i \sigma_i$, and their respective Lagrange multipliers. The final result for the exponent $f = \frac{1}{N} \ln\langle \mathcal{N}_q \rangle$, obtained via a saddle-point integration, is

$$\begin{aligned} f = q\tilde{q} + A_x\tilde{A}_x + A_y\tilde{A}_y + B_x\tilde{B}_x + B_y\tilde{B}_y + C_x\tilde{C}_x + C_y\tilde{C}_y + D\tilde{D} \\ + \alpha(C_x + C_y) + \alpha G_1(A_x, A_y, B_x, B_y, C_x, C_y, D) \end{aligned}$$

$$+G_0(\tilde{q}, \tilde{A}_x, \tilde{A}_y, \tilde{B}_x, \tilde{B}_y, \tilde{C}_x, \tilde{C}_y, \tilde{D}) \tag{6}$$

where

$$G_1 = \ln \int \frac{\mathrm{d}m\, \mathrm{d}\tilde{m}}{2\pi} \int \frac{\mathrm{d}n\, \mathrm{d}\tilde{n}}{2\pi} \exp[-\tfrac{1}{2}A_x n^2 - \tfrac{1}{2}A_y m^2 - Dmn] \exp[-\tfrac{1}{2}\tilde{m}^2 - \tfrac{1}{2}\tilde{n}^2 - q\tilde{m}\tilde{n}] \\ \times \exp[\mathrm{i}\tilde{m}(B_x n + C_y m) + \mathrm{i}\tilde{n}(B_y m + C_x n)] \tag{7}$$

and

$$G_0 = \ln \sum_S \sum_\sigma \exp(-\tilde{q}S\sigma) \int \frac{\mathrm{d}x\, \mathrm{d}\tilde{x}}{2\pi} \Theta(x) \int \frac{\mathrm{d}y\, \mathrm{d}\tilde{y}}{2\pi} \Theta(y) \exp[-\tilde{A}_x \tilde{x}^2 - \tilde{A}_y \tilde{y}^2] \\ \times \exp[-\tilde{D}\tilde{x}\tilde{y}S\sigma - \mathrm{i}\tilde{x}(\tilde{B}_x + \tilde{C}_x S\sigma) - \mathrm{i}\tilde{y}(\tilde{B}_y + \tilde{C}_y S\sigma)]. \tag{8}$$

The saddle-point parameters and their Lagrange multipliers are determined so as to maximize f. To proceed further we assume the symmetric ansatz $A_x = A_y = A$, and similarly for the other saddle-point parameters, which corresponds to the sensible strategy of leaving the symmetry $S \leftrightarrow \sigma$ intact. With this ansatz the evaluation of G_1 and G_0 is greatly facilitated, yielding

$$G_1 = -\tfrac{1}{2} \ln\{(A^2 - D^2)(1 - q^2) + [B^2 - (1 + C)^2] \\ -4B(1 + C)(D + qA) + 2[B^2 + (1 + C)^2](A + qD)\} \tag{9}$$

and

$$G_0 = \ln \left\{ \mathrm{e}^{-\tilde{q}} \int_{-(\tilde{B}+\tilde{C})/2\tilde{A}^{1/2}}^{\infty} \mathrm{D}z\, \mathrm{erfc}\left[\frac{-\tilde{A}^{1/2}(\tilde{B} + \tilde{C}) - z\tilde{D}}{(4\tilde{A}^2 - \tilde{D}^2)^{1/2}} \right] \right. \\ \left. +\mathrm{e}^{\tilde{q}} \int_{-(\tilde{B}-\tilde{C})/2\tilde{A}^{1/2}}^{\infty} \mathrm{D}z\, \mathrm{erfc}\left[\frac{-\tilde{A}^{1/2}(\tilde{B} - \tilde{C}) + z\tilde{D}}{(4\tilde{A}^2 - \tilde{D}^2)^{1/2}} \right] \right\} \tag{10}$$

where $\mathrm{D}z = \mathrm{d}z/\sqrt{\pi}\mathrm{e}^{-z^2}$ is the Gaussian measure. We are left then with the highly non-trivial numerical problem of maximizing the function

$$f = q\tilde{q} + 2A\tilde{A} + 2B\tilde{B} + 2C(\tilde{C} + \alpha) + D\tilde{D} + \alpha G_1(A, B, C, D) + G_0(\tilde{q}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \tag{11}$$

with respect to seven parameters. Fortunately, an appropriate redefinition of these parameters allows us to reduce the problem to the solution of three coupled saddle-point equations only.

The result of the numerical maximization of equation (11) is presented in figure 1, which shows the exponent f as a function of the storage parameter α for several values of q. For the sake of clarity we present only the region of positive values of f. We note that, for a given q, a negative f indicates that for almost all realizations of the stored patterns that type of cycle is absent. In particular, for $\alpha = 0$ we find $f = \ln 2$ for $q = -1$, $f = 0$ for $q = 1$, and $f \to -\infty$ for all other values of q. Moreover, except for the cases $q = \pm 1$, the curves present a maximum which, due to the range of α used in figure 1, is perceptible for $q = -0.95$ only. For $\alpha \to \infty$ all curves tend to the result obtained for the SK model (Sherrington and Kirkpatrick 1975), namely, $f \approx 0.1992$ (Bray and Moore 1980). For small α the inverse cycles ($q = -1$) are the dominant ones, but for $\alpha \approx 0.7$ other cycles with $|q| < 1$ become dominant. A systematic study of the value of q that maximizes f for a given α was not possible due to the extreme unwieldliness of the system of saddle-point equations.

A rather unexpected, but not difficult to understand, outcome of our analysis is the dominance of the inverse cycles for small α. For simplicity, let us consider the infinite

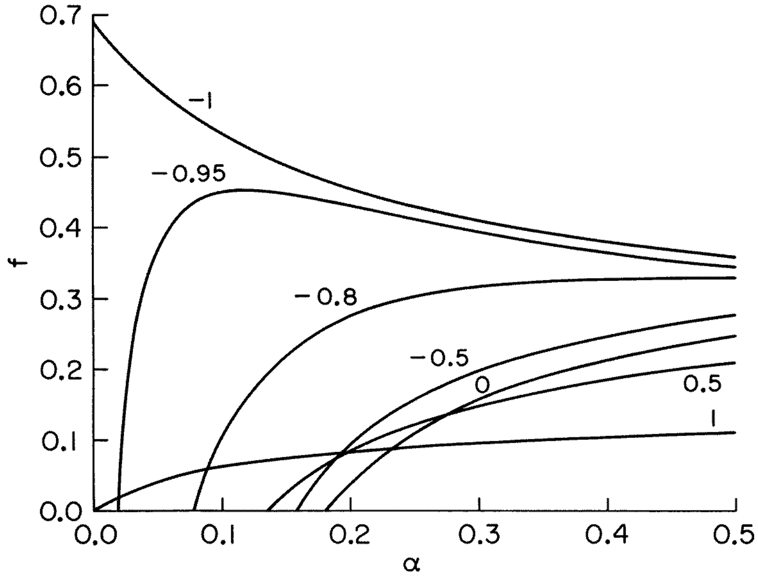

Figure 1. The exponent f in $\langle \mathcal{N}_q \rangle = \exp(Nf)$ as a function of the storage parameter α for $q = -1$ (inverse cycles), -0.95, -0.8, -0.5, 0, 0.5 and 1 (fixed points).

range ferromagnetic Ising model, whose configuration space is dominated by states with zero magnetization $\sum_i S_i = 0$. Given one such state, half of the spins will be stable and half unstable. The simultaneous flipping of the unstable spins will very likely unstabilize the other half. The process is then repeated leading to an inverse cycle.

In summary, we have extended the general analytical method of counting the number of metastable states (fixed points) in order to take into account cyclic attractors as well. The technique applied to the Little model has shown that the cycles of length two are by far the most common type of attractors. Thus, if the design of associative memories aims at minimizing the number of spurious attractors, the sequential dynamics must be the preferred one. It would be interesting to apply this method to investigate analytically the effects of random dilution and asymmetry in the length of the attractors for a simpler model as, for instance, the synchronous SK model (Gutfreund *et al* 1988).

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